Mathematics

Example

An ellipse has a semimajor axis with length $a = 12$ and a semiminor axis with length $b = 3$. What is the approximate length of the perimeter of the ellipse?

(A) 24
(B) 47
(C) 55
(D) 180

Solution

Use Eq. 2.78.

$$P_{\text{approx}} = 2\pi\sqrt{(a^2 + b^2)/2} = 2\pi\sqrt{\frac{12^2 + 3^2}{2}}$$
$$= 54.96 \quad (55)$$

The answer is (C).

Equation 2.81 and Eq. 2.82: Circular Segments

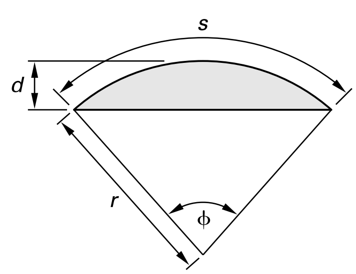
$$A = [r^2(\phi - \sin\phi)]/2 \qquad 2.81$$

$$\phi = s/r = 2\{\arccos[(r - d)/r]\} \qquad 2.82$$

Description

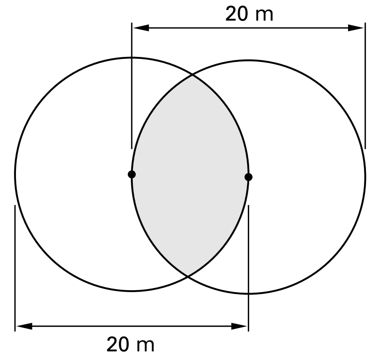
A *circular segment* is a region bounded by a circular arc and a chord, as shown by the shaded portion in Fig. 2.18. The arc and chord are both limited by a central angle, ϕ. Use Eq. 2.81 to find the area of a circular segment when its central angle, ϕ, and the radius of the circle, r, are known; in Eq. 2.81, the central angle must be in radians. Use Eq. 2.82 to find the central angle when the radius of the circle and either the height of the circular segment, d, or the length of the arc, s, are known.

Figure 2.18 *Circular Segment*



Example

Two 20 m diameter circles are placed so that the circumference of each just touches the center of the other.



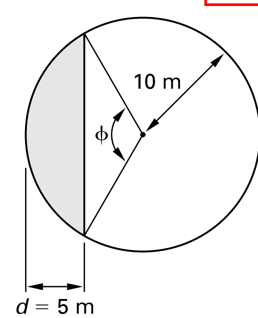
What is most nearly the area of the shared region?

(A) 62 m^2
(B) 110 m^2
(C) 120 m^2
(D) 170 m^2

Solution

The shared region can be thought of as two equal circular segments, each as shown in the illustration. The radius of each circle is $r = 10$ m. The height of each circular segment is half the radius, so $d = 6$ m.

d = 5 m.



Use Eq. 2.82 to find the angle ϕ.

$$\phi = 2\{\arccos[(r - d)/r]\} = 2\arccos\left(\frac{10\ \text{m} - 5\ \text{m}}{10\ \text{m}}\right)$$
$$= 120°$$

Convert ϕ to radians.

$$\phi = (120°)\left(\frac{2\pi}{360°}\right) = 2.094\ \text{rad}$$

Equation 3.15 and Eq. 3.16: Converting from Rectangular Form to Polar Form

$$r = |x + jy| = \sqrt{x^2 + y^2} \quad 3.15$$

$$\theta = \arctan(y/x) \quad 3.16$$

Description

The polar form of a complex number, $r(\cos\theta + j\sin\theta)$, can be determined from the complex number's rectangular coordinates x and y using Eq. 3.15 and Eq. 3.16.

Example

The rectangular coordinates of a complex number are $(4, 6)$. What are the complex number's approximate polar coordinates?

(A) $(4.0, 33°)$

(B) $(4.0, 56°)$

(C) $(7.2, 33°)$

(D) $(7.2, 56°)$

Solution

The radius and angle of the polar form can be determined from the x- and y-coordinates using Eq. 3.15 and Eq. 3.16.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (6)^2} \\ &= 7.211 \quad (7.2) \\ \theta &= \arctan(y/x) = \arctan\frac{6}{4} \\ &= 56.3° \quad (56°) \end{aligned}$$

The answer is (D).

Equation 3.17 and Eq. 3.18: Multiplication and Division with Polar Forms

$$\begin{aligned} &[r_1(\cos\theta_1 + j\sin\theta_1)][r_2(\cos\theta_2 + j\sin\theta_2)] \\ &\quad = r_1 r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)] \end{aligned} \quad 3.17$$

$$\frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} = \frac{r_1}{r_2}\begin{bmatrix} \cos(\theta_1 - \theta_2) \\ + j\sin(\theta_1 - \theta_2) \end{bmatrix} \quad 3.18$$

Variations

$$z_1 z_2 = (r_1 r_2)\angle(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Description

The multiplication and division rules defined for complex numbers expressed in rectangular form can be applied to complex numbers expressed in polar form. Using the trigonometric identities, these rules reduce to Eq. 3.17 and Eq. 3.18.

Equation 3.19: de Moivre's Formula

$$\begin{aligned} (x + jy)^n &= [r(\cos\theta + j\sin\theta)]^n \\ &= r^n(\cos n\theta + j\sin n\theta) \end{aligned} \quad 3.19$$

Description

Equation 3.19 is *de Moivre's formula*. This equation is valid for any real number x and integer n.

Equation 3.20 Through Eq. 3.23: Euler's Equations

$$e^{j\theta} = \cos\theta + j\sin\theta \quad 3.20$$

$$e^{-j\theta} = \cos\theta - j\sin\theta \quad 3.21$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad 3.22$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad 3.23$$

Description

Complex numbers can also be expressed in exponential form. The relationship of the exponential form to the trigonometric form is given by *Euler's equations*, also known as *Euler's identities*.

Example

If $j = \sqrt{-1}$, which of the following is equal to j^j?

(A) j^2

(B) e^{2j}

(C) -1

(D) $e^{-\pi/2}$

Note: For non-integer values of n, de Moivre's formula produces one possible solution.

Solution

j is the imaginary unit vector, so $r = 1$ and $\theta = 90°\ (\pi/2)$ in Fig. 3.1. From Eq. 3.19,

$$(j)^n = (\cos\theta + j\sin\theta)^n$$

Description

As indicated in Eq. 3.42, the dot product of two parallel unit vectors is one.

Example

What is the dot product $\mathbf{A}\cdot\mathbf{B}$ of unit vectors $\mathbf{A} = 3\mathbf{i}$ and $\mathbf{B} = 2\mathbf{i}$?

(A) –6

(B) –5

(C) 5

(D) 6

Solution

Use Eq. 3.42.

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= 3\mathbf{i}\cdot 2\mathbf{i} = (3\cdot 2)\mathbf{i}\cdot\mathbf{i} = (6)(1)\\ &= 6\end{aligned}$$

The answer is (D).

Equation 3.43: Dot Product of Orthogonal Vectors

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 0 \qquad 3.43$$

i . j = j . k = k . i = 0

Description

The dot product can be used to determine whether a vector is a unit vector and to show that two vectors are orthogonal (perpendicular). As indicated in Eq. 3.43, the dot product of two non-null (nonzero) orthogonal vectors is zero.

Equation 3.44 Through Eq. 3.46: Cross Product Identities

$$\mathbf{A}\cdot\ \mathbf{B} = -\mathbf{B}\cdot\ \mathbf{A} \qquad 3.44$$

$$\mathbf{A}\cdot\ (\mathbf{B}+\mathbf{C}) = (\mathbf{A}\cdot\ \mathbf{B}) + (\mathbf{A}\cdot\ \mathbf{C}) \qquad 3.45$$

$$(\mathbf{B}+\mathbf{C})\cdot\ \mathbf{A} = (\mathbf{B}\cdot\ \mathbf{A}) + (\mathbf{C}\cdot\ \mathbf{A}) \qquad 3.46$$

Description

The vector cross product is distributive, as demonstrated in Eq. 3.45 and Eq. 3.46. However, as Eq. 3.44 shows, it is not commutative.

Equation 3.47: Cross Product of Parallel Unit Vectors

$$\mathbf{i}\cdot\ \mathbf{i} = \mathbf{j}\cdot\ \mathbf{j} = \mathbf{k}\cdot\ \mathbf{k} = 0 \qquad 3.47$$

Description

If two non-null vectors are parallel, their cross product will be zero.

Equation 3.48 and Eq. 3.49: Cross Product of Normal Unit Vectors

$$\mathbf{i}\cdot\ \mathbf{j} = \mathbf{k} = -\mathbf{j}\cdot\ \mathbf{i} \qquad 3.48$$

$$\mathbf{k}\cdot\ \mathbf{i} = \mathbf{j} = -\mathbf{i}\cdot\ \mathbf{k} \qquad 3.49$$

Description

If two non-null vectors are normal (perpendicular), their vector cross product will be perpendicular to both vectors.

10. PROGRESSIONS AND SERIES

A *progression* or *sequence*, $\{\mathbf{A}\}$, is an ordered set of numbers a_i, such as 1, 4, 9, 16, 25, . . . The *terms* in a sequence can be all positive, negative, or of alternating signs. l is the last term and is also known as the *general term* of the sequence.

$$\{\mathbf{A}\} = a_1, a_2, a_3, \ldots, l$$

A sequence is said to *diverge* (i.e., be *divergent*) if the terms approach infinity, and it is said to *converge* (i.e., be *convergent*) if the terms approach any finite value (including zero).

A *series* is the sum of terms in a sequence. There are two types of series: A *finite series* has a finite number of terms. An *infinite series* has an infinite number of terms, but this does not imply that the sum is infinite. The main tasks associated with series are determining the sum of the terms and determining whether the series converges. A series is said to converge if the sum, S_n, of its terms exists. A finite series is always convergent. An infinite series may be convergent.

Equation 3.50 and Eq. 3.51: Arithmetic Progression

$$l = a + (n-1)d \qquad 3.50$$

$$S = n(a+l)/2 = n[2a + (n-1)d]/2 \qquad 3.51$$

Description

The *arithmetic progression* is a standard sequence that diverges. It has the form shown in Eq. 3.50.

In Eq. 3.50 and Eq. 3.51, a is the *first term*, d is a constant called the *common difference*, and n is the number of terms.

The characteristic equation is Eq. 5.8.

Depending on the form of the forcing function, the solutions to most second-order differential equations will contain sinusoidal terms (corresponding to oscillatory behavior) and exponential terms (corresponding to decaying or increasing unstable behavior). Behavior of real-world systems (electrical circuits, spring-mass-dashpot, fluid flow, heat transfer, etc.) depends on the amount of system *damping* (electrical resistance, mechanical friction, pressure drop, thermal insulation, etc.).

With *underdamping* (i.e., with "light" damping) without continued energy input (i.e., a free system without a forcing function), the transient behavior will gradually decay to the steady-state equilibrium condition. Behavior in underdamped free systems will be oscillatory with diminishing magnitude. The damping is known as underdamping because the amount of damping is less than the critical damping, and the *damping ratio*, ζ, is less than 1. The characteristic equation of underdamped systems has two complex roots.

With *overdamping* ("heavy" damping), damping is greater than critical, and the damping ratio is greater than 1. Transient behavior is a sluggish gradual decrease into the steady-state equilibrium condition without oscillations. The characteristic equation of overdamped systems has two distinct real roots (zeros).

With *critical damping*, the damping ratio is equal to 1. There is no overshoot, and the behavior reaches the steady-state equilibrium condition the fastest of the three cases, without oscillations. The characteristic equation of critically damped systems has two identical real roots (zeros).

Equation 5.9 Through Eq. 5.14: Roots of the Characteristic Equation

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \qquad 5.9$$

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \qquad 5.10$$

$$y = (C_1 + C_2 x) e^{r_1 x} \qquad 5.11$$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \qquad 5.12$$

$$\alpha = -a/2 \qquad 5.13$$

$$\beta = \frac{\sqrt{4b - a^2}}{2} \qquad 5.14$$

Description

The roots of the characteristic equation are given by the quadratic equation, Eq. 5.9.

If $a^2 > 4b$, then the two roots are real and different, and the solution is overdamped, as shown in Eq. 5.10.

If $a^2 = 4b$, then the two roots are real and the same (i.e., are *double roots*), and the solution is critically damped, as shown in Eq. 5.11.

If $a^2 < 4b$, then the two roots are imaginary and of the form $(\alpha + i\beta)$ and $(\alpha - i\beta)$, and the solution is underdamped, as shown in Eq. 5.12.

Example

What is the general solution to the following homogeneous differential equation?

$$y'' - 8y' + 16y = 0$$

(A) $y = C_1 e^{4x}$

(B) $y = (C_1 + C_2 x) e^{4x}$

(C) $y = C_1 e^{-4x} + C_2 e^{4x}$

(D) $y = C_1 e^{2x} + C_2 e^{4x}$

Solution

Find the roots of the characteristic equation.

$$\begin{aligned} r^2 - 8r + 16 &= 0 \\ a &= -8 \\ b &= 16 \end{aligned}$$

From Eq. 5.9,

REMOVE THIS "2"

$$\begin{aligned} r_{1,2} &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \\ &= \frac{-(-8) \pm 2\sqrt{(-8)^2 - (4)(16)}}{2} \\ &= 4, 4 \end{aligned}$$

Because $a^2 = 4b$, the characteristic equation has double roots. With $r = 4$, the solution takes the form

$$\begin{aligned} y &= (C_1 + C_2 x) e^{rx} \\ &= (C_1 + C_2 x) e^{4x} \end{aligned}$$

The answer is (B).

Table 5.5: Laplace Transform Pairs

***Table 5.5** Laplace Transforms*

$f(t)$	$F(s)$
$\delta(t)$, impulse at $t = 0$	1
$u(t)$, step at $t = 0$	$1/s$
$t[u(t)]$, ramp at $t = 0$	$1/s^2$
$e^{-\alpha t}$	$1/(s+\alpha)$
$te^{-\alpha t}$	$1/(s+\alpha)^2$
$e^{-\alpha t}\sin\beta t$	$\beta/\left[(s+\alpha)^2+\beta^2\right]$
$e^{-\alpha t}\cos\beta t$	$(s+\alpha)/[(s+\alpha)^2+\beta^2]$
$\dfrac{d^n f(t)}{dt^n}$	$s^nF(s) - \sum_{m=0}^{n-1} s^{n-m-1}\dfrac{d^m f(0)}{dt^m}$
$\int_0^t f(\tau)d\tau$	$(1/s)F(s)$
$\int_0^t x(t-\tau)h(t)d\tau$	$H(s)X(s)$
$f(t-\tau)u(t-\tau)$	$e^{-\tau s}F(s)$

Description

Table 5.5 gives common Laplace transforms.

Example

What is the Laplace transform of the step function $f(t)$?

$$f(t) = u(t-1) + u(t-2)$$

(A) $\dfrac{1}{s} + \dfrac{2}{s}$

(B) $\dfrac{e^{-s} + e^{-2s}}{s}$

(C) $1 + \dfrac{e^{-2s}}{s}$

(D) $\dfrac{e^{s}}{s} + \dfrac{e^{2s}}{s}$

Solution

The notations $u(t-1)$ and $u(t-2)$ mean that a unit step input (a step of height 1) is applied at $t = 1$, and another unit step is applied at $t = 2$. (This function could be used to describe the terrain that a tracked robot would have to navigate to go up a flight of two stairs in a particular interval.) Table 5.5 contains Laplace transforms for various input functions, including steps. For steps at $t = 0$, the Laplace transform is $1/s$. However, in this example, the steps are encountered at $t = 1$ and $t = 2$. Superposition can be used to calculate the Laplace transform of the summation as the sum of the two transforms. Use the last entry in Table 4.5, [Table 5.5] with $f(t-\tau)=1$.

$$F(s) = F\big(u(t-1)\big) + F\big(u(t-2)\big) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$= \frac{e^{-s} + e^{-2s}}{s}$$

The answer is (B).

Equation 5.34: Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}dt \qquad 5.34$$

Description

Extracting a function from its transform is the *inverse Laplace transform* operation. Although Eq. 5.34 could be used and other methods exist, this operation is almost always done using a table, such as Table 5.5.

Equation 5.35: Initial Value Theorem

$$\lim_{s\to\infty} sF(s) \qquad 5.35$$

Description

Equation 5.35 shows the *initial value theorem* (IVT).

Equation 5.36: Final Value Theorem

$$\lim_{s\to 0} sF(s) \qquad 5.36$$

Description

Equation 5.36 shows the *final value theorem* (FVT).

7. DIFFERENCE EQUATIONS

Equation 5.37: Difference Equation

$$f(t) = y' = \frac{y_{i+1} - y_i}{t_{i+1} - t_i} \qquad 5.37$$

Description

Many processes can be accurately modeled by differential equations. However, exact solutions to these models may be difficult to obtain. In such cases, discrete versions of the original differential equations can be produced. These discrete equations are known as *finite difference equations* or just *difference equations*.

The probability of picking a white ball first and an orange ball second is

$$\begin{aligned} P(A,B) &= P(A)P(B|A) \\ &= \left(\frac{2}{17}\right)\left(\frac{7}{16}\right) \\ &= 0.05147 \quad (0.052) \end{aligned}$$

(B)

The answer is (D).

Equation 6.12: Bayes' Theorem

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)} \qquad 6.12$$

Variation

$$P(B_j|A) = \frac{P(B \text{ and } A)}{P(A)}$$

Description

Given two dependent sets of events, A and B, the probability that event B will occur given the fact that the dependent event A has already occurred is written as $P(B_j|A)$ and is given by *Bayes' theorem*, Eq. 6.12.

Example

A medical patient exhibits a symptom that occurs naturally 10% of the time in all people. The symptom is also exhibited by all patients who have a particular disease. The incidence of that particular disease among all people is 0.0002%. What is the probability of the patient having that particular disease?

(A) 0.002%

(B) 0.01%

(C) 0.3%

(D) 4%

Solution

This problem is asking for a conditional probability: the probability that a person has a disease, D, given that the person has a symptom, S. Use Bayes' theorem to calculate the probability. The probability that a person has the symptom S given that they have the disease D is $P(S|D)$ and is 100%. Multiply by 100% to get the answer as a percentage.

$$\begin{aligned} P(D|S) &= \frac{P(D)P(S|D)}{P(S|D)P(D) + P(S \mid \text{not } D)P(\text{not } D)} \\ &= \frac{(0.000002)(1.00)}{(1.00)(0.000002) + (0.10)(0.999998)} \\ &= 0.00002 \quad (0.002\%) \end{aligned}$$

The answer is (A).

4. MEASURES OF CENTRAL TENDENCY

It is often unnecessary to present experimental data in their entirety, either in tabular or graphic form. In such cases, the data and distribution can be represented by various parameters. One type of parameter is a measure of *central tendency*. The mode, median, and mean are measures of central tendency.

Mode

The *mode* is the observed value that occurs most frequently. The mode may vary greatly between series of observations; its main use is as a quick measure of the central value, since little or no computation is required to find it. Beyond this, the usefulness of the mode is limited.

Median

The *median* is the point in the distribution that partitions the total set of observations into two parts containing equal numbers of observations. It is not influenced by the extremity of scores on either side of the distribution. The median is found by counting from either end through an ordered set of data until half of the observations have been accounted for. If the number of data points is odd, the median will be the exact middle value. If the number of data points is even, the median will be the average of the middle two values.

Equation 6.13: Arithmetic Mean

$$\overline{X} = (1/n)(X_1 + X_2 + \cdots + X_n) = (1/n)\sum_{i=1}^{n} X_i \qquad 6.13$$

Variation

$$\overline{X} = \frac{\sum f_i X_i}{\sum f_i} \quad \left[\begin{array}{c} f_i \text{ are frequencies} \\ \text{of occurrence of} \\ \text{events } i \end{array}\right]$$

Example

Samples of aluminum-alloy channels were tested for stiffness. The following distribution of results was obtained.

stiffness	frequency
2480	23
2440	35
2400	40
2360	33
2320	21

2400

If the mean of the samples is 2402, what is the approximate standard deviation of the population from which the samples are taken?

(A) 48.2

(B) 49.7

(C) 50.6

(D) 50.8

Solution

The number of samples is

$$n = 23 + 35 + 40 + 33 + 21 = 152$$

The sample standard deviation, s, is the unbiased estimator of the population standard deviation, σ.

$$\begin{aligned} s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2} \\ &= \sqrt{\left(\frac{1}{152-1}\right)\left(\begin{array}{l}(23)(2480-2402)^2 \\ \quad +(35)(2440-2402)^2 \\ \quad +(40)(2400-2402)^2 \\ \quad +(33)(2360-2402)^2 \\ \quad +(21)(2320-2402)^2\end{array}\right)} \\ &= 50.82 \quad (50.8) \end{aligned}$$

2400 2400 2400 2400 2400

50.85

The answer is (D).

Equation 6.23 Through Eq. 6.25: Variance and Sample Variance

$$\sigma^2 = (1/N)[(X_1 - \mu)^2 + (X_2 - \mu)^2 + \cdots + (X_N - \mu)^2] \qquad 6.23$$

$$\sigma^2 = (1/N)\sum_{i=1}^{N}(X_i - \mu)^2 \qquad 6.24$$

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2 \qquad 6.25$$

Description

The *variance* is the square of the standard deviation. Since there are two standard deviations, there are two variances. The *variance of the population* (i.e., the *population variance*) is σ^2, and the *sample variance* is s^2. The population variance can be found using either Eq. 6.23 or Eq. 6.24, both derived from Eq. 6.17, and the sample variance can be found using Eq. 6.25, derived from Eq. 6.22.

Example

Most nearly, what is the sample variance of the following data set?

$$2, 4, 6, 8, 10, 12, 14$$

(A) 4.3

(B) 5.2

(C) 8.0

(D) 19

Solution

Find the mean using Eq. 6.13.

$$\begin{aligned} \overline{X} &= (1/n)\sum_{i=1}^{n}X_i = \left(\frac{1}{7}\right)(2+4+6+8+10+12+14) \\ &= 8 \end{aligned}$$

From Eq. 6.25, the sample variance is

$$\begin{aligned} s^2 &= [1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2 \\ &= \left(\frac{1}{7-1}\right)\left(\begin{array}{l}(2-8)^2 + (4-8)^2 + (6-8)^2 \\ \quad +(8-8)^2 + (10-8)^2 + (12-8)^2 \\ \quad +(14-8)^2\end{array}\right) \\ &= 18.67 \quad (19) \end{aligned}$$

The answer is (D).

Description

Although sand will experience its consolidation (and settlement) almost instantly after a load is applied, clayey soils approach their eventual consolidations gradually. Equation 18.38 is used to calculate the settlement at a particular moment in time after the soil is loaded. U_{AV} is the average *degree of consolidation.*

S_{ULT} is the settlement calculated from Eq. 18.37. Usually, the degree of consolidation is calculated from the time factor, which in turn, depends on the time, layer thickness, and the *coefficient of consolidation.*[33]

Equation 18.39: Degree of Consolidation Time Factor

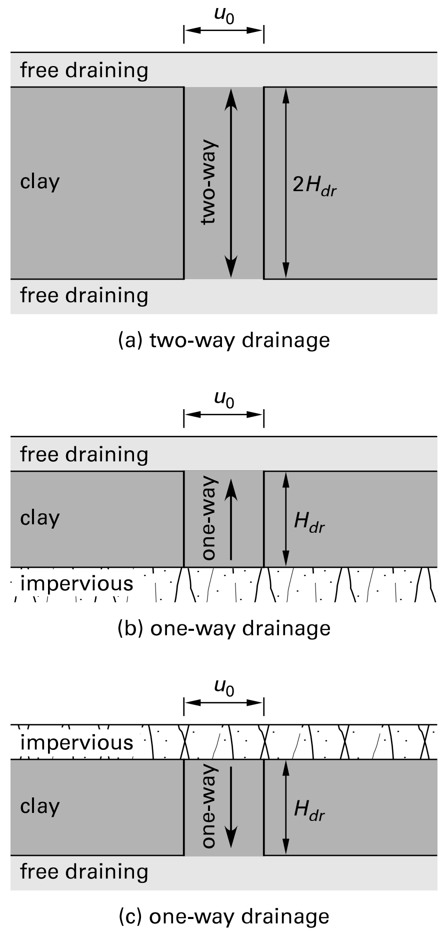
$$T_v = \frac{c_v t}{H_{dr}^2} \qquad 18.39$$

Description

The degree of consolidation, U, is the percentage of the total settlement of a clay layer achieved at a time, t. The degree of consolidation can be determined by dividing the change in excess pore pressure at time t by the initial excess pore pressure, u_0. The initial excess pore pressure is constant with the depth of the soil layer.

The time factor, T_v, given by Eq. 18.39, is a function of the average degree of consolidation, U (%), of a saturated clay layer at time t. Table 18.6 shows the variation of the time factor, T_v, with the degree of consolidation. The coefficient of consolidation, c_v, has units of square distance per time. The length of the average maximum drainage path, H_{dr}, is dependent on the soil type above and below the clay layer. Figure 18.12 shows the three types of drainage path conditions with a constant pore pressure u_0. Figure 18.12(a) depicts two-way drainage, with free-draining soil above and below the clay layer; in this case, the height of drainage, H_{dr}, is one-half the full height of the clay layer, with half the excess pore pressure both above and below the soil. Figure 18.12(b) and Fig. 18.12(c) depict one-way drainage, where either the soil layer above or below the clay layer are impervious; in these cases, the height of drainage is the full height of the clay layer.

Figure 18.12 Different types of Drainage with u_0 constant



Example

A 10 ft thick clay layer has a permeable sand layer above and below it. The coefficient of consolidation is 0.25 ft^2/day. Most nearly, what is the rate of consolidation?

(A) 125 days

(B) 153 days

(C) 187 days

(D) 191 days

55
68
81
85

Solution

Sand above and below the clay layer indicates two-way drainage, so the height of drainage, H_{dr}, is cut in half. From Table 18.6, for a degree of consolidation, U, of

[33] t_c is not related to the preconsolidation pressure, p_c, although they share the same subscript. t_c is the time since the load was applied. T is used to designate the time factor, although the traditional symbol is T_v.

Table 18.6 *Variation of Time Factor with Degree of Consolidation**

U (%)	T_v	U (%)	T_v	U (%)	T_v
0	0	34	0.0907	68	0.377
1	0.00008	35	0.0962	69	0.390
2	0.0003	36	0.102	70	0.403
3	0.00071	37	0.107	71	0.417
4	0.00126	38	0.113	72	0.431
5	0.00196	39	0.119	73	0.446
6	0.00283	40	0.126	74	0.461
7	0.00385	41	0.132	75	0.477
8	0.00502	42	0.138	76	0.493
9	0.00636	43	0.145	77	0.511
10	0.00785	44	0.152	78	0.529
11	0.0095	45	0.159	78	0.547
12	0.0113	46	0.166	80	0.567
13	0.0133	47	0.173	81	0.588
14	0.0154	48	0.181	82	0.610
15	0.0177	49	0.188	83	0.633
16	0.0201	50	0.197	84	0.658
17	0.0227	51	0.204	85	0.684
18	0.0254	52	0.212	86	0.712
19	0.0283	53	0.221	87	0.742
20	0.0314	54	0.230	88	0.774
21	0.0346	55	0.239	89	0.809
22	0.0380	56	0.248	90	0.848
23	0.0415	57	0.257	91	0.891
24	0.0452	58	0.267	92	0.938
25	0.0491	59	0.276	93	0.993
26	0.0531	60	0.286	94	1.055
27	0.0572	61	0.297	95	1.129
28	0.0615	62	0.307	96	1.219
29	0.0660	63	0.318	97	1.336
30	0.0707	64	0.329	98	1.500
31	0.0754	65	0.340	99	1.781
32	0.0803	66	0.352	100	∞
33	0.0855	67	0.364		

*u_0 constant with depth

90%, the time factor, T_v, is 0.848. The rate of consolidation is

$$T_v = \frac{c_v t}{H_{dr}^2}$$

$$t = \frac{H_{dr}^2 T_v}{c_v}$$

$$= \frac{(0.75\text{ ft})^2(0.848)}{0.25\ \frac{\text{ft}^2}{\text{day}}}$$

$$= 190.8 \text{ days} \quad (191 \text{ days})$$

5 ft

85

84.8

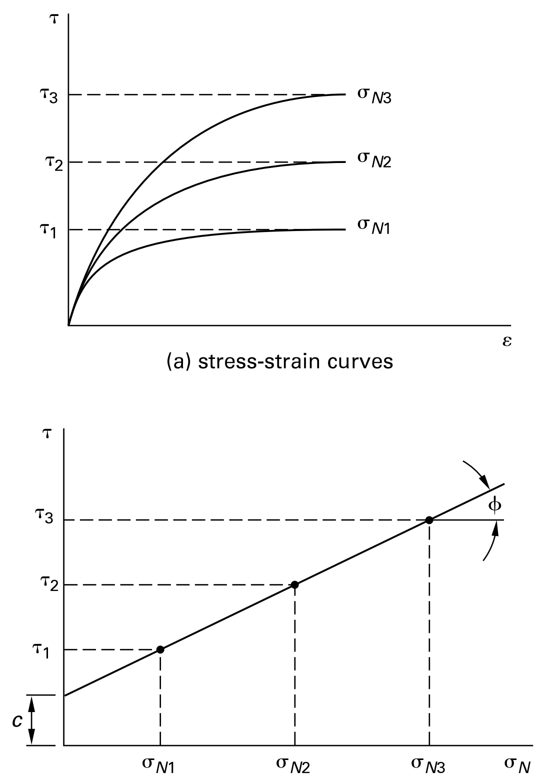
The answer is (D).

20. DIRECT SHEAR TEST

The *direct shear test* is a relatively simple test used to determine the relationship of shear strength to consolidation stress. In this test, a disc of soil is inserted into the direct shear box. The box has a top half and a bottom half that can slide laterally with respect to each other. A normal stress, σ_N, is applied vertically, and then one half of the box is moved laterally relative to the other at a constant rate. Measurements of vertical and horizontal displacement, δ, and horizontal shear load, T, are taken. The test is usually repeated at three different vertical normal stresses. (See Fig. 18.13.)

Because of the box configuration, failure is forced to occur on a horizontal plane. Results from each test are plotted as horizontal displacement versus horizontal *shear stress*, τ (horizontal force divided by the nominal area). *Failure* is determined as the maximum value of horizontal stress achieved. The vertical normal stress and failure stress from each test are then plotted in Mohr's circle space of normal stress versus shear stress.

Figure 18.13 *Graphing Direct-Shear Test Results*



(a) stress-strain curves

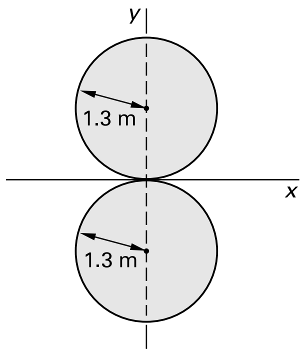
(b) Mohr's failure envelope

A line drawn through all of the test values is called the *failure envelope* (*failure line* or *rupture line*).

Geotechnical Engineering

Example

For the composite plane area made up of two circles as shown, the moment of inertia about the y-axis is 4.7 cm^4, and the moment of inertia about the x-axis is 23.5 cm^4.



What is the approximate polar moment of inertia of the area taken about the intersection of the x- and y-axes?

(A) 0 cm^4

(B) 14 cm^4

(C) 28 cm^4

(D) 34 cm^4

Solution

Use the perpendicular axis theorem, as given by Eq. 25.171.

$$\begin{aligned} J &= I_y + I_x \\ &= 4.7\ \text{cm}^4 + 23.5\ \text{cm}^4 \\ &= 28.2\ \text{cm}^4 \quad (28\ \text{cm}^4) \end{aligned}$$

The answer is (C).

Equation 25.173 and Eq. 25.174: Parallel Axis Theorem

$$I'_x = I_{x_c} + d_x^2 A \qquad 25.173$$

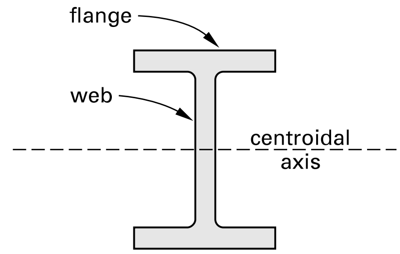
$$I'_y = I_{y_c} + d_y^2 A \qquad 25.174$$

Description

If the moment of inertia is known with respect to one axis, the moment of inertia with respect to another, parallel axis can be calculated from the *parallel axis theorem*, also known as the *transfer axis theorem*. This theorem is used to evaluate the moment of inertia of areas that are composed of two or more basic shapes. d is the distance between the centroidal axis and the second, parallel axis.

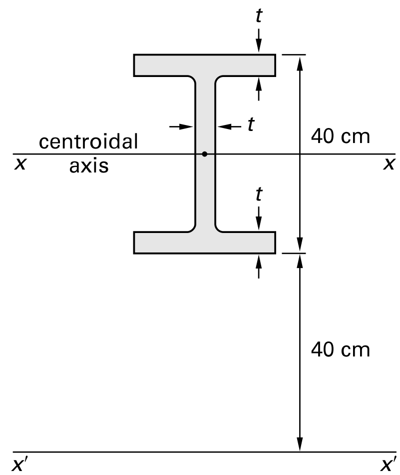
The second term in Eq. 25.173 and Eq. 25.174 is often much larger than the first term in each equation, since areas close to the centroidal axis do not affect the moment of inertia considerably. This principle is exploited in the design of structural steel shapes that derive bending resistance from *flanges* located far from the centroidal axis. The *web* does not contribute significantly to the moment of inertia. (See Fig. 25.1.)

Figure 25.1 *Structural Steel Shape*



Example

The moment of inertia about the x'-axis of the cross section shown is 334 000 cm^4. The cross-sectional area is 86 cm^2, and the thicknesses of the web and the flanges are the same.



What is most nearly the moment of inertia about the centroidal axis?

(A) 2.4×10^4 cm^4

(B) 7.4×10^4 cm^4

(C) 2.0×10^5 cm^4

(D) 6.4×10^5 cm^4

Statics

Example

15 minute interval traffic counts for a road are shown. Most nearly, what is the peak hour factor?

time	vehicles
4:00–4:15 pm	130
4:15–4:30 pm	120
4:30–4:45 pm	140
4:45–5:00 pm	150
5:00–5:15 pm	160
5:15–5:30 pm	140
5:30–5:45 pm	170
5:45–6:00 pm	180

(A) 0.50

(B) 0.65

(C) 0.80

(D) 0.90

Solution

Tabulate the hourly volumes.

time	hourly volume (vehicles)
4:00–5:00 pm	540
4:15–5:15 pm	570
4:30–5:30 pm	590
4:45–5:45 pm	620
5:00–6:00 pm	650

The peak hour occurs between 5:00 pm and 6:00 pm, with a peak hour volume of 650 vehicles. The peak 15 minute volume within the peak hour is 180 vehicles. From Eq. 48.10, the peak hour factor is

$$\begin{aligned}\text{PHF} &= \frac{\text{hourly volume}}{4 \times \text{peak 15 minute volume}} \\ &= \frac{650 \text{ vehicles}}{(4)(\cancel{170} \text{ vehicles})} \\ &= 0.903 \quad (0.90)\end{aligned}$$

180

The answer is (D).

Equation 48.11 Through Eq. 48.14: Traffic Speed and Density Relationships

$$S = S_f - \frac{S_f}{D_j} D \qquad \text{48.11}$$

$$V = S_f D - \frac{S_f}{D_j} D^2 \qquad \text{48.12}$$

$$V_m = \frac{D_j S_f}{4} \qquad \text{48.13}$$

$$D_o = \frac{D_j}{2} \qquad \text{48.14}$$

Description

Traffic models are used to investigate the relationship when one parameter is changed with respect to another. The most important traffic model is the one developed between speed and density. Greenshields proposed a linear speed-density relationship, representing the first and most simple of the speed and density models. This relationship, often referred to as the *Greenshields' model*, is given by Eq. 48.11, where S is the mean speed at density D, S_f is the free flow speed, and D_j is the *jam density*. The Greenshields' model demonstrates that as density approaches zero, speed approaches free flow.

Knowing that flow, V, is equal to the product of speed and density, the relationship between flow and density can be derived. Substituting this relationship into Eq. 48.11 creates a parabolic model, given by Eq. 48.12. The maximum flow, V_m, given by Eq. 48.13 is one-fourth the product of free flow speed and jam density. The optimum density at maximum flow, D_o, given by Eq. 48.14, is sometimes called the *critical density*, and is expressed as half of the jam density.

Example

A data set for a freeway segment shows a free flow speed of 55 mi/hr and a jam density of 150 veh/mi. The observed density of the segment is 75 veh/mi. Most nearly, what is the travel speed?

(A) 28 mi/hr

(B) 35 mi/hr

(C) 50 mi/hr

(D) 55 mi/hr

The number of times the instructions are executed depends on when the condition is no longer true. The variable or variables that control the condition must eventually be changed by the operations, or the WHILE loop will continue forever.

DO/UNTIL loops: A set of instructions between the DO/UNTIL <condition> and the ENDUNTIL lines of code is repeated as long as the condition remains false. The number of times the instructions are executed depends on when the condition is no longer false. The variable or variables that control the condition must eventually be changed by the operations, or the UNTIL loop will continue forever.

FOR loops: A set of instructions between the FOR <counter range> and the NEXT <counter> lines of code is repeated for a fixed number of loops that depends on the counter range. The counter is a variable that can be used in operations in the loop, but the value of the counter is not changed by anything in the loop besides the NEXT <counter> statement.

GOTO: A GOTO operation moves the program to a number designator elsewhere on the program. The GOTO statement has fallen from favor and is avoided whenever possible in structured programming.

Example

A computer structured programming segment contains the following program segment.

```
Set G = 1 and X = 0
DO WHILE G ≤ 5
    X = G*X + 1
    X = G
ENDWHILE
```

G = X

What is the value of G after the segment is executed?

(A) 5

(B) 26

(C) 63

(D) The loop never ends.

Solution

The first execution of the WHILE loop results in

$$G = (1)(0) + 1 = 1$$
$$X = 1$$

The second execution of the WHILE loop results in

$$G = (1)(1) + 1 = 2$$
$$X = 2$$

The third execution of the WHILE loop results in

$$G = (2)(2) + 1 = 5$$
$$X = 5$$

The WHILE condition is still satisfied, so the instruction is executed a fourth time.

$$G = (5)(5) + 1 = 26$$
$$X = 26$$

The answer is (B).

9. HIERARCHY OF OPERATIONS

Operations in an arithmetic statement are performed in the order of exponentiation first, multiplication and division second, and addition and subtraction third. In the event that there are two consecutive operations with the same hierarchy (e.g., a multiplication followed by a division), the operations are performed in the order encountered, normally left to right (except for exponentiation, which is right to left).[1] Parentheses can modify this order; operations within parentheses are always evaluated before operations outside. If nested parentheses are present in an expression, the expression is evaluated outward starting from the innermost pair.

Example

Based on standard processing and hierarchy of operations, what would most nearly be the result of the following calculation?

$$459 + 306\ /\ (6*5\hat{}2 + 3)$$

(A) 3

(B) 5

(C) 459

(D) 461

Solution

The standard mathematical processing hierarchy is represented by the acronym "BODMAS": Brackets (parentheses), Order (exponentiation or powers), Division and

[1]In most implementations, a statement will be scanned from left to right. Once a left-to-right scan is complete, some implementations then scan from right to left; others return to the equals sign and start a second left-to-right scan. Parentheses should be used to define the intended order of operations.